Topic 6 – Second order linear ODEs Theory

We will now consider second order linear ODEs of the form

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = b(x)$$

where $a_2(x), a_1(x), a_0(x), b(x)$ are continuous on some interval I.

For now we will assume that $a_2(x) \neq 0$ for all x in I

If $a_2(x) = 0$ at some point x in I then you get a "singular point" and that needs other techniques.

We may also include initial-value constraints

$$y'(x_0) = y'_0 \quad \text{and} \quad y(x_0) = y_0$$

where x_0 is in the interval I.

Ex: Consider the ODE

$$y'' - 7y' + 10y = 24e^{x} \qquad (*)$$

on the interval $I = (-\infty, \infty)$

Let

$$f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

note that f is defined on all of I

where c_1, c_2 are any constants.

We will now show that f solves $(*)$ on I.

We have that

$$f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$
$$f'(x) = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$
$$f''(x) = 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^{x}$$

Thus, plugging f into the LHS of $(*)$ gives

$$f''(x) - 7f'(x) + 10f(x)$$

$$= 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^{x}$$
$$- 14c_1 e^{2x} - 35c_2 e^{5x} - 42e^{x}$$
$$+ 10c_1 e^{2x} + 10c_2 e^{5x} + 60e^{x}$$

$$= 24e^{x}$$

Thus, f solves $(*)$ on I.

Now let's see if we can use f to get a solution to the initial-value problem

$$\left.\begin{aligned} y'' - 7y' + 10y &= 24e^{x} \\ y'(0) = 6, \quad y(0) &= 0 \end{aligned}\right\} \quad (**)$$

on the interval $I = (-\infty, \infty)$

We know that $f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$ solves $y'' - 7y' + 10y = 24e^{x}$.

Let's see if we can find c_1, c_2 so that $f'(0) = 6$, $f(0) = 0$

$$6 = 2c_1 e^{0} + 5c_2 e^{0} + 6e^{0} \quad \leftarrow \quad 6 = f'(0)$$

$$0 = c_1 e^{0} + c_2 e^{0} + 6e^{0} \quad \leftarrow \quad 0 = f(0)$$

This gives

$$\boxed{\begin{aligned} 0 &= 2c_1 + 5c_2 \\ -6 &= c_1 + c_2 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

We get $c_1 = -c_2 - 6$ from 2.

Plug this into 1 to get $0 = 2(-c_2-6)+5c_2$.

So, $12 = 3c_2$.

So, $c_2 = 4$.

Then, $c_1 = -4-6 = -10$.

Thus, $f(x) = -10e^{2x} + 4e^{5x} + 6x$ solves the initial-value problem (**)

Summary of the above

The function $f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^x$ is a solution to $y'' - 7y' + 10y = 24e^x$ for any constants c_1, c_2.

If we further impose the restriction that $y'(0) = 6$ and $y(0) = 0$ then

$$f(x) = -10e^{2x} + 4e^{5x} + 6x$$

solves the ODE.

For the remainder of the class we will work on developing different methods to solve second order ODEs.

Below we have a theorem for linear second order ODEs on when solutions exist and are unique.

<u>Theorem</u>: Let I be an interval. Let $a_2(x), a_1(x), a_0(x), b(x)$ be continuous on I and $a_2(x) \neq 0$ for all x in I. If x_0 is a fixed point in I, then the initial-value problem

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

$$y'(x_0) = y_0', \quad y(x_0) = y_0$$

has a unique solution on I.

We will begin with solving the homogeneous linear second order ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

To do this we need to learn about linear independence.

<u>Def:</u> Let I be an interval. Let f_1, f_2 be functions defined on I. We say that f_1 and f_2 are <u>linearly dependent on I</u> if one of them is a multiple of the other on I, that is if either

$$f_2(x) = c_1 f_1(x) \quad \text{for all } x \text{ in } I$$

or

$$f_1(x) = c_2 f_2(x) \quad \text{for all } x \text{ in } I$$

where c_1, c_2 are constants.

If no such constants exist, then we say that f_1 and f_2 are <u>linearly independent</u> on I.

<u>Ex:</u> Let $I = (-\infty, \infty)$.

Let $f_1(x) = x^2$ and $f_2(x) = -5x^2$.

Then f_1 and f_2 are linearly dependent on I since

$$f_2(x) = -5 f_1(x)$$

for all x in I.

Or you could say that

$$f_1(x) = -\frac{1}{5} f_2(x)$$

for all x in I.

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$.

We will show that f_1 and f_2 are linearly independent on I.

We must show that f_1, f_2 are not multiples of each other on I.

Case 1: Let's show that f_2 is not a multiple of f_1 on I.

Suppose $f_2(x) = c_1 f_1(x)$ for all x in I.

Then, $e^{5x} = c_1 e^{2x}$ for all x in $(-\infty, \infty)$.

So,

$$e^0 = c_1 e^0 \qquad x = 0$$

$$e^5 = c_1 e^2 \qquad x = 1$$

Thus,

$$1 = c_1$$

$$e^3 = c_2$$

But this is a contradiction since $1 \neq e^3$. Thus, no such c_1 exists.

So, f_2 is not a multiple of f_1 on I.

<u>case 2:</u> Can f_1 be a multiple of f_2 on I?

Suppose $f_1(x) = c_2 f_2(x)$ for all x in I.

Then, $e^{2x} = c_2 e^{5x}$ for all x in $(-\infty, \infty)$.

So,

$$e^0 = c_2 e^0 \quad \leftarrow x = 0$$

$$e^2 = c_2 e^5 \quad \leftarrow x = 1$$

Then,

$$1 = c_2$$

$$e^{-3} = c_2$$

But this can't happen since $1 \neq e^{-3}$.

So, f_1 cannot be a multiple of f_2.

By case 1 and case 2 we know that f_1 and f_2 are linearly independent on I.

We will now give a faster way to check for linear independence on an interval I. It uses a method called the Wronskian (named after Jusef Wronski 1778–1853).

<u>Def</u>: Let f_1 and f_2 be differentiable functions on an interval I. The <u>Wronskian</u> of f_1 and f_2 is the following determinant

$$W(f_1, f_2) = \underbrace{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}_{\text{notation for determinant}} = f_1(x)f_2'(x) - f_2(x)f_1(x)$$

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

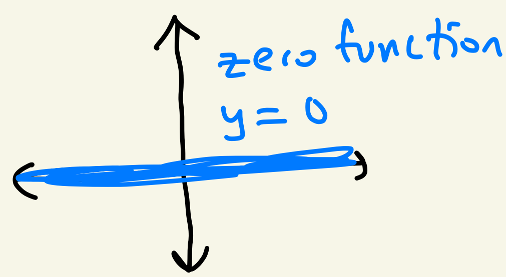
Ex: The Wronskian of $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$ is

$$W(e^{2x}, e^{5x}) = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

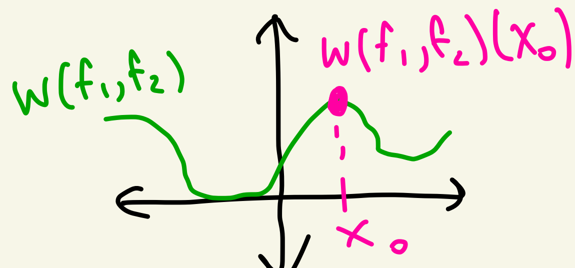
$$= (e^{2x})(5e^{5x}) - (e^{5x})(2e^{2x})$$

$$= 3e^{7x}$$

Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian $W(f_1, f_2)$ is not the zero function on I, then f_1 and f_2 are linearly independent on I.



That is, if there exists some x_0 in I where $W(f_1, f_2)(x_0) \neq 0$, then f_1 and f_2 are linearly independent.



Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.

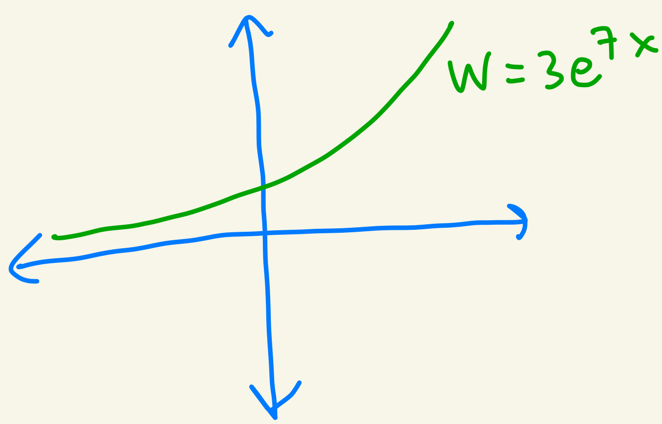
Let's show that f_1 and f_2 are linearly independent.

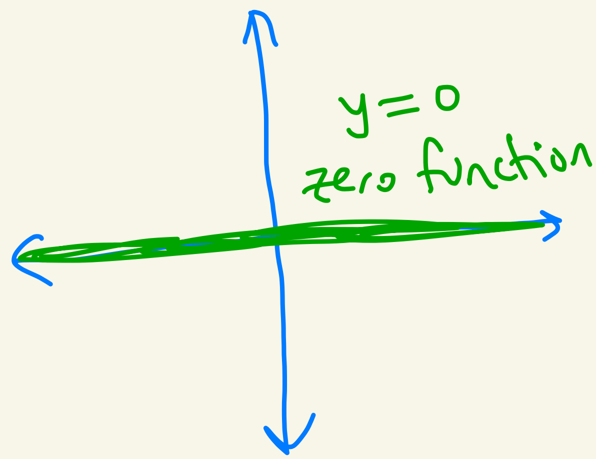
Above we saw that

$$W(f_1, f_2) = 3e^{7x}$$

We want to find some x_0 in $I = (-\infty, \infty)$ where $W(f_1, f_2) = 3e^{7x_0}$ is not zero

side note: What we are doing is checking whether or not $3e^{7x}$ is the zero function





For example at $x_0 = 0$ we see that

$$W(f_1, f_2)(0) = 3e^{7(0)} = 3e^0 = 3 \neq 0$$

Thus by the previous theorem $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent on $I = (-\infty, \infty)$

Theorem: [Linear, homogeneous, second order ODE]

Let I be an interval.

Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \qquad (***)$$

Suppose that

- $f_1(x)$ and $f_2(x)$ are linearly independent on I, and
- $f_1(x)$ and $f_2(x)$ are both solutions to $(***)$

Then every solution to $(***)$ is of the form

$$y_h = c_1 f_1(x) + c_2 f_2(x)$$

for some constants c_1, c_2.

we will call this y_h
h is for homogeneous ie the $=0$ ODE

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.

We saw above that f_1 and f_2 are linearly independent on I.

Note that f_1 and f_2 both solve the homogeneous, linear, second order ODE

$$y'' - 7y' + 10y = 0$$

Check:

$f_1(x) = e^{2x}$, $f_1'(x) = 2e^{2x}$, $f_1''(x) = 4e^{2x}$

$f_1'' - 7f_1' + 10f_1 = 4e^{2x} - 14e^{2x} + 10e^{2x} = 0$

$f_2(x) = e^{5x}$, $f_2'(x) = 5e^{5x}$, $f_2''(x) = 25e^{5x}$

$f_2'' - 7f_2' + 10f_2 = 25e^{5x} - 35e^{5x} + 10e^{5x} = 0$

Therefore the general solution to

$$y'' - 7y' + 10y = 0$$

is

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

Now we look at the general second order linear ODE.

Theorem: Let I be an interval.

Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

Suppose that f_1 and f_2 are linearly independent solutions to the homogeneous eqn

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

on I.

Suppose that y_p is a particular solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

on I.

Then every solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

is of the form

$$f(x) = y_h + y_p = \overbrace{c_1 f_1(x) + c_2 f_2(x)}^{y_h} + \overbrace{y_p(x)}^{y_p}$$

for some constants c_1, c_2.

Ex: Consider the ODE

$$y'' - 7y' + 10y = 24e^{x}$$

on the interval $I = (-\infty, \infty)$

We saw earlier that $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent solutions to

$$y'' - 7y' + 10y = 0$$

and so $\boxed{y_h = c_1 e^{2x} + c_2 e^{5x}}$

y_h is general solution to this homogeneous eqn.

Let $y_p = 6e^{x}$. Then y_p is a particular solution to $y'' - 7y' + 10y = 24e^{x}$ since:

$$y_p(x) = 6e^{x}$$

$$y_p'(x) = 6e^{x}$$

$$y_p''(x) = 6e^{x}$$

Plugging y_p into $y'' - 7y' + 10y = 0$ we see it solves the equation:

$$\overbrace{y_p'' - 7y_p' + 10}^{y'' - 7y' + 10} = 6e^{x} - 42e^{x} + 60e^{x} = \overbrace{24e^{x}}^{24e^{x}}$$

Thus, by our theorems, every solution to

$$y'' - 7y' + 10y = 24e^{x}$$

is of the form

$$f(x) = y_h + y_p = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

Now our goal is to answer these questions:

1. How do we find two linearly independent solutions to the homogeneous equation
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

2. How do we find a particular solution to
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

We will work on these problems over the next several lessons.

The following are proofs of some of the previus theorems for those that are interested.
We won't cover this in class
It's mostly for me :)
You would need some linear algebra and proofs background to read.

Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian $W(f_1, f_2)$ is not zero for at least one point in I, then f_1 and f_2 are linearly independent on I.

proof:
Suppose f_1 and f_2 are linearly dependent on I.
Then there exist c_1, c_2, not both zero, where

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x in I.
Thus,

$$c_1 f_1'(x) + c_2 f_2'(x) = 0$$

for all x in I.
So,
$$\begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we get that $\begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix}$ is not invertible for each x in I.
Thus, $W(f_1, f_2)(x) = 0$ for all x in I. $\blacksquare$

Theorem: [Linear, homogeneous, second order ODE]

Let I be an interval.
Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0 \qquad (***)$$

Suppose that

- $f_1(x)$ and $f_2(x)$ are linearly independent on I, and
- $f_1(x)$ and $f_2(x)$ are both solutions to $(***)$

Then every solution to $(***)$ is of the form

$$c_1 f_1(x) + c_2 f_2(x)$$

later we will call this y_h

for some constants c_1, c_2.

proof:

By linearity, $c_1 f_1(x) + c_2 f_2(x)$ will be a solution to $(***)$.
Since f_1 and f_2 are linearly independent

on I, by the previous theorem there exists t in I where $W(f_1,f_2)(t) \neq 0$.

Let Ψ be some solution of $(***)$.

Consider the system

$$c_1 f_1(t) + c_2 f_2(t) = \Psi(t)$$
$$c_1 f_1'(t) + c_2 f_2'(t) = \Psi'(t)$$

This system will have a unique solution for c_1, c_2 since

$$W(f_1,f_2)(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} \neq 0.$$

Let $\hat{c}_1, \hat{c}_2$ be the unique solution and define

$$Z(t) = c_1 f_1(x) + c_2 f_2(x).$$

By the linearity of $(***)$ we know Z satisfies $(***)$. Z also satisfies the initial conditions $Z(t) = \Psi(t)$, $Z'(t) = \Psi'(t)$ from above. Since Ψ satisfies the same initial value problem, by the uniqueness theorem we have $\Psi(x) = Z(x)$ for all x in I. $\square$

Theorem: Let I be an interval.

Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

Suppose that f_1 and f_2 are linearly independent solutions to the homogeneous eqn

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

on I.

Suppose that f_p is a particular solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

on I.

Then every solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

is of the form

$$f(x) = \overbrace{c_1 f_1(x) + c_2 f_2(x)}^{y_h} + f_p(x)$$

for some constants c_1, c_2.

proof:

Let f solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$. Then, $f - f_p$ will solve the homogeneous equation. Hence $f - f_p = c_1 f_1 + c_2 f_2$ for some c_1, c_2. So, $f = c_1 f_1 + c_2 f_2 + f_p$ ☐